Analysis of Fluid (potential) flow by Complex Conjugate method

Mr. Fekadu Yemataw Sitotaw

Department of Mathematics, College of Natural and Computational Science Wollega University Wollega, Ethiopia.

Abstract— The method of modeling the fluid flow around two or three dimention requires an understanding of compl ex mathematical analysis. The harmonic conjugate method is the main focus of this paper. This paper is geared tow ards scholars with an understanding of multivariable calculus who wish to develop their understanding of complex var iable mathematics . In complex analysis the harmonic conjugate method is a useful intermediate step that allows for complicated two or three dimention flow problems to be solved as problems with simpler than working in real. The str eam function ψ applies both for rotational and irrotational flows. However the potential function ϕ is applicable only for irrotational flow.

Key Terms: harmonic conjugate, stream function, potential function

1. Introduction

In the field of fuid dynamics, the study of flued flows, t he equation of continuity is the Mathematical expres sion for the principle of conservation of mass flow [1]. A streamline is defined as an imaginary line drawn thr ough a flow field such that the tangent to the line at a ny point on the line indicates the direction of the veloc ity vector at that instant. A streamline thus gives a pict ure of the average direction of flow in a flow field. If a series of streamlines are drawn through every point o n the perimeter of a small area of a stream cross-secti on, they will form a stream tube. Since there is no flo w across a streamline, there will not be flow across a stream tube and the fluid inside a stream tube cannot escape through its walls. The flow thus behaves as if i t were contained in an imaginary pipe. The concept of a stream tube is useful in dealing with the flow of fluid s since it allows elements of the fluid to be isolated for analysis [1].

Understanding fluid flow at a higher level thus involve s the physical modeling of the fuid.While this physical modeling is obviously an area of interest to those stud ying introduct-ory fuid dynamics, it is also a relevant fi eld for scholars studying advanced mathematics. In p articular, one common method of modeling the fluid flow around two or three dimention requires an understanding of complex mathematical analysis. This method, known as harmonic conjugate, will be the main focus of this paper. My paper is therefore geared towards scholars with an understanding of multivariable calculus who wish to develop their understanding of complex variable mathematics.

Harmonic conjugate is a mathematical technique in which complicated real valued function can be obtained by a transforming in to function complex variables [3]. Using this te-chnique, the fluid flow around the two demintional geometry of can be analyzed. In this analysis, we focus on modeling the two-dimensional fluid flow.

2. Overview of Theories Used in Mathematical Modeling

2.1 The Cauchy Riemann equations and harmonic conjugates

Definition: (Functions of a complex variable)

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each z in S a complex number w. The number w is called the value of at z denoted by f(z), that is, $f(z) = w$.

The set S is called the domain of the function f.

Suppose $z = x + iy$ and $w = u + iv$ so that $f(z) = u(x, y) + iv(x, y)$

and it follows that f(z) can be expressed in terms of a pair of real-valued functions (u &v) of the real variabls

Theorem. Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

And that $\boldsymbol{f}'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the fist-order partial derivatives of $\boldsymbol{u}$ and v must exist

at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$(1) \ldots\ldots\ldots\ldots u_x = v_y \text{ , and } v_x = -u_y$$

There. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x$$

Where these partial derivatives are to be evaluated at (x_0, y_0)

Proof; We start by writing $z_0 = x_0 + iy_0$, $\Delta z = \Delta x + i\Delta y$, and

$$\Delta w = f(z_0 + \Delta z) - f(z_0)$$

$$\Delta w = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]$$

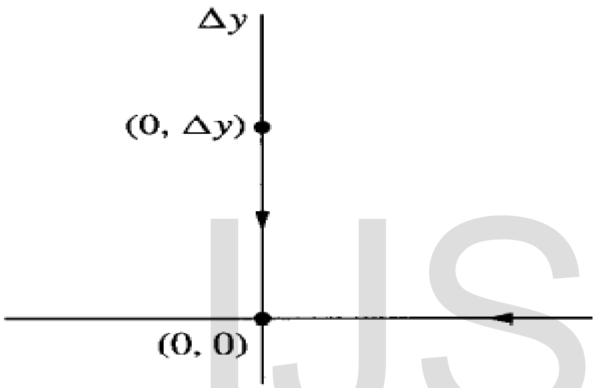
Assuming that the derivative

$$(2) \ldots\ldots\ldots\ldots f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta W}{\Delta z}$$

exists, we know from Theorem 1 of definition of derivatives that

$$(3)\ldots\ldots f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} Re[\frac{\Delta W}{\Delta z}] + i \lim_{(\Delta x,\Delta y)\to(0,0)} Im[\frac{\Delta W}{\Delta z}]$$

Now it is important to keep in mind that expression **(3)** is valid as $(\Delta x, \Delta y)$ tends to $(0, 0)$ in any manner that we may choose. In particular, we let $(\Delta x, \Delta y)$ tends to $(0, 0)$ horizontally through the points $(\Delta x, 0)$, as indicated in Fig. below.



Inasmuch as $\Delta y = 0$, the quotient $\frac{\Delta W}{\Delta z}$ becomes

$$\frac{\Delta W}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i\,\frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

Thus,

$$\lim_{(\Delta x,\Delta y)\to(0,0)} Re[\frac{\Delta W}{\Delta z}] = \lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} = u_x(x_0, y_0),$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} Im[\frac{\Delta W}{\Delta z}] = \lim_{\Delta x\to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} = v_x(x_0, y_0),$$

Where $u_x(x_0, y_0)$ and $u_x(x_0, y_0)$ denote the first-order partial derivatives with respect to x of the functions *u* and *v*, respectively, at (x_0, y_0). Substitution of these limits into expression **(3)** tells us that

$$(4)\ldots\ldots\ldots f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0)$$

We might have let Δz tend to zero vertically through the points $(0, \Delta y)$. In this case, $\Delta x =$ 0 and the quotient $\frac{\Delta W}{\Delta z}$ becomes

$$\frac{\Delta W}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i\,\frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$\frac{\Delta W}{\Delta z} = \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - i\,\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$$

Thus,

$$\lim_{(\Delta x,\Delta y)\to(0,0)} Re[\frac{\Delta W}{\Delta z}] = -\lim_{\Delta y\to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} = -u_y(x_0, y_0),$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} Im[\frac{\Delta W}{\Delta z}] = \lim_{\Delta y\to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta x} = v_y(x_0, y_0),$$

Hence it follows from expression **(3)** that

$$(5)\ldots\ldots\ldots f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0)$$

Where $u_y(x_0, y_0)$ and $u_y(x_0, y_0)$ denote the first-order partial derivatives with respect to y of the functio ns *u* and *v*, respectively, at (x_0, y_0). Note that equation *(5)* can also be written in the form

$$f'(z_0) = -i[u_y(x_0, y_0) + iv_y(x_0, y_0)]$$

Equations (4) and (5) not only give $f'(z_0)$ in terms of partial derivatives of the component functions u *and* v, but they also provide necessary conditions for the existence of $f'(z_0)$. For, on equating the real and imaginary parts on the right-hand sides of these equations, we see that the existence of $f'(z_0)$ requires that

$$(6)\ldots.\ u_x(x_0, y_0) = v_y(x_0, y_0)\ ,\ and\ v_x(x_0, y_0) = -u_y(x_0, y_0)$$

Equations (6) are the *Cauchy-Riemann equations,*

Harmonic Functions: A real-valued function F of two real variables x and y is said to be **harmonic** in a given domain of the xy-plane if, throughout that domain, it has a continuous partial derivatives of the first and second order and satisfies the partial differential equation (Laplace's equation.)

$$F_{xx}(x, y) + F_{yy}(x, y) = 0$$

For example $F(x, y) = e^{-y} \sin x$ is harmonic function in the xy-plane.

Harmonic Conjugate: If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy-Riemann equation throughout D, v is said to be **harmonic conjugate** of u.

Example 11: Show that the function $u(x, y) = y^3 - 3x^2y$ is harmonic and find its harmonic conjugatev(x, y).

Solution: $u_x = -6xy$, $u_{xx} = -6y$ and $u_y = 3(y^2 - x^2)$, $u_{yy} = 6y$, then $u_{xx} + u_{yy} = 0$. Therefore $u(x, y)$ is harmonic. To find its harmonic conjugate by means of the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. The first equation tells us that $v_y(x, y) = -6xy$ and then integrating with and then integrating with respect to y, holding x fixed we get $v(x, y) = -3xy^2 + h(x)$. Differentiating this w.r.t x we get $v_x = -3y^2 + h'(x)$. From $u_y = -v_x$ we can have

$$3(y^2 - x^2) = 3y^2 - h'(x) \text{or} \quad h'(x) = 3x^2$$
$$\Longrightarrow h(x) = x^3 + c$$

Therefore $v(x, y) = -3xy^2 + x^3 + c$ is the harmonic conjugate of $u(x, y)$, c is any real constant.

The corresponding analytic function is

$$f(z) = (y^3 - 3x^2y) + i(-3xy^2 + x^3 + c)$$

2.2 Stream Function

The rate of flow becomes a function. This function is known as stream function and denoted by ψ. If the value of ψ is constant along a streamline. Each streamline will have a different value of ψ

such that the difference in ψ values of two streamlines gives the flow rate between the two streamlines. The relationship between the velocity components in the x and y directions of a two dimensional flow and the stream function ψ developed by considering the two streamlines. Note that what is required is the difference in the value of the stream functions between two streamlines, and not the absolute value of ψ, in the determination of the velocity of the discharge.

Hence the value $\psi = 0$ can be assigned to any streamline.

Since $\psi = \psi(x, y)$ and $d\psi = \frac{\partial \psi}{\partial x}\delta x + \frac{\partial \psi}{\partial y}\delta y$

Taking the velocity of flow across δ_y to be u and that across δ_x to be $-\boldsymbol{v}$, it is clear that: Comparing this equation with the above total differential $d\psi$, it is clear that:

$$u = \frac{\partial \psi}{\partial x} \text{ and } v = -\frac{\partial \psi}{\partial y} \qquad \text{------------------------ 2.2.1}$$

Equations 2.2.1 is the relationship between the stream function in the x-y plane and the velocity components in the x and y directions. Examination of the continuity equation for two-dimensional incompressible flow;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

in light of the relations expressed in equation 2.1 and substitution gives:

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0$$

which shows that the continuity equation is identically satisfied. Therefore, the existence of a stream function ψ for a flow implies a possible flow and conversely, for any possible flow, a stream function ψ must exist. Considering the equation of a streamline in the x-y plane;

$$\frac{dx}{u} = \frac{dy}{v} \text{ or } udy - vdx = 0$$

Substituting the values of u and ψ from equation 2. 2.1, The left hand side of the above equation is the total differential $d\psi$ of $\psi = f(x, y)$ Since, $d\psi = 0$, $\psi = c$ constant along a streamline.

2.3 velocity potential

Analogous to the principle that electric current flows in the direction of decreasing voltage and that the rate of flow of current is proportional to the difference in voltage potential between two points, the velocity of flow of a fluid in a particular direction would depend on certain potential difference called velocity potential [2]. The velocity potential, denoted by ϕ(phi), decreases in the direction of flow. It has no absolute value and is simply a scalar function of position and time. For steady flow, the velocity components u, v and w in the x, y and z directions respectively, in terms of velocity potential are:

$$u = -\frac{\partial \phi}{\partial x} \text{ , } v = \frac{\partial \phi}{\partial y} \text{ and } w = -\frac{\partial \phi}{\partial z} \text{ --------2.3.1}$$

A potential line is a line along which the potential ϕ is constant. Thus if a potential function exists for a certain flow, then it is possible to draw lines of constant potential. Some of the properties of the potential function may be derived by substituting it in the equations of rotation and continuity. Substituting the values of the velocity components u, v and w of equation 2.3.1 in the expression for rotation, one obtains:

$$\omega_x = \frac{1}{2}[\frac{\partial w}{\partial z} - \frac{\partial v}{\partial z}] = \frac{1}{2}[\frac{\partial^2 \phi}{\partial z \partial y} - \frac{\partial^2 \phi}{\partial y \partial z}]$$

$$\omega_y = \frac{1}{2}[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}] = \frac{1}{2}[\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}]$$

$$\omega_z = \frac{1}{2}[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}] = \frac{1}{2}[\frac{\partial^2 \phi}{\partial y \partial x} - \frac{\partial^2 \phi}{\partial x \partial y}]$$

If ϕ is a continuous function, $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$

etc. Hence $w_x = w_y = w_z = 0$

and, which is the condition for irrotationality [2]. Therefore, if a velocity potential ϕ

exists, then the flow should be irrotational and vice versa. Substitution of the velocity components given in equation **2.3.1** in the three dimensional continuity equation leads to the Laplace Equation:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \text{ -----------------2.3.2}$$

Hence, any function ϕ

which satisfies the Laplace Equation is a case of steady, incompressible, irrotational flow and such a flow is known as ***potential flow.*** It should be noted that the stream function ψ

applies both for rotational and irrotational flows. However the potential function ϕ

is applicable only for irrotational flow. Equations 2.2.1 and **2.3.1** may be used to establish the relationship between stream function ψ and potential function ϕ for an irrotational, steady, incompressible flow leading to the following:

$$\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \text{ ----------------------------2.3.2}$$

Equations **2.3.2** are known as the Cauchy-Riemann Equations. Hence the stream function ψ

and the potential function ϕ

are conjugate each other.

and the complex valued function $f(z) = \psi + i\phi$

analytic function.

Example: We now illustrate one method of obtaining a harmonic conjugate of a given harmonic function. Th

6

e function let the stream function $\psi = y^3 - 3x^2 y$ find the potential function ϕ ?

Solution: let $\psi = u(x, y)$ and $\phi = v(x, y)$

$$(1) \dots\dots\dots\dots\dots\; u(x, y) = y^3 - 3x^2 y$$

is readily seen to be harmonic throughout the entire xy plane. Since a harmonic conjugate $v(x, y)$ is rela ted to $u(x , y)$ by means of the Cauchy-Riemann eq uations

$$(2) \dots\dots\dots\dots\; u_x = v_y \text{ and } u_y = -v_x$$

the first of these equations tells us that

$$u_x = v_y = -6xy$$

Holding x fixed and integrating each side here with re spect to y, we find that

$$(3) \dots\dots\dots\; v(x, y) = -3xy^2 + \varphi(x)$$

where $\varphi(x)$ is, at present, an arbitrary function of x.

Using the second of equations ***(3),*** we have

$$3y^2 - 3x^2 = 3y^2 - \varphi(x)$$

or $\varphi(x) = 3x^2$. Thus $\varphi(x) = x^3 + c$, where c is an arbitrary real number. According to equation (7), then, the function

$$(4) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\; v(x, y) = -3xy^2 + x^3 + c$$

is a harmonic conjugate of $u\,(x\,, y)$.

The corresponding analytic function is

$$(4) \dots f(z) = y^3 - 3x^2 y + i[-3xy^2 + x^3 + c]$$

The form $f\,(z) = i\,(z^3 + c)$ of this function is easily verified and is suggested by noting that when $y = 0$, expression (5) becomes $f\,(x) = i\,(x^3 + c)$.

Thus the stream function $\psi = y^3 - 3x^2 y$

then the potential function $\phi = x^3 - 3xy^2 + c$

3. Conclusion

The paper presents a complex conjugate method of c alculating the stream function ψ and potential function ϕ around two and three dimensional. The following con clusions can be drawn from the present complex anal ysis:

(i) The present method can be an efficient tool for eva luating the stream function ψ and potential function ϕ of two dimensional body.

(ii) The real and imaginary parts of an analytic functio n satistfies the Laplace's equation in two variables. H ence two dimens-ional potential problems can be solv ed by methods for ananlytic functions and this is simpl er than working in real.

(iii) Study of analytic function leads to a deeper of the stream function ψ and potential function ϕ properties and to interrelations in complex understa nding that have no analog in real calculus.

References

[1] Erwin Kreysing (20006); "Advanced Engineering Mathematics", Ohio State University; Columbus, Ohio, 9th Ed.

[2] Dr. Solomon Alemu (1992) ; " Essentials of Hydraulics part-I", Addis Abeba University Press; Ethiopia

[3] James Ward Brown and Ruel V. Churchil l (20090; " Complex Variables and Its Application"; Library of Congres s Cataloging-in-Publication Data, 8th Ed.

[4] Ahlfors, L. V. (1979); "Complex Analysis,", McGraw-Hill Higher Education, Burr Ridge, IL, 3rd Ed..

[5] Bak, J., and D. J. Newman (1997); "Complex Analysis," 2d ed., Springer-Verlag, New York,2nd Ed.

[6] Bieberbach, L. (2000); "Conformal Mapping," American Mathematical Society, Providence, RI.

[7] Boas, R. P. (1987); "Invitation to Complex Analysis," The McGraw-Hill Companies, New York.

[8] Dake, J.M.K.(, 1983), "Essentials of Engineering Hydraulics", The Macmillan Press, London, 2nd Ed.

[9] Daugherty, R.L., and Franzini, J.B.(1965), "Fluid Mechanics with Engineering Applications", McGRaw Hill.

[10] Douglas, J.F., Gasiorek, J.A., and Swaffield, J.A. (1985), "Fluid Mechanics", Longman.

[11] Garde, R.J., and Mirajgacker, A.G.(1977), "Engineering Fluid Mechanics", NemChand and Bros., Roorkee.

[12] Kumar, K.L. (1988), "Engineering Fluid Mechanics", Eurasia Publishing House, New Delhi.

[13] Massey, B.S. (1970) , "Mechanics of Fluids", Van. Nostrand Ranhold Company.

[14] Nagaratnam, S.(1975), "Fluid Mechanics", Khanna Publishers, New Delhi.